LEWIS
GRANT
IN-34-CR
NAG 3-~~1048~~
NAG 3 1208
1581
P. 11

Solution of the Navier-Stokes Equations for a Driven Cavity

B.D. Semeraro A. Sameh

March 26, 1991

1 Introduction

The flow field in a lid driven cavity is determined by integration of the incompressible Navier-Stokes equations. The numerical integration is accomplished via an operator splitting method known as the Θ-scheme. (see Appendix for a description of the Θ scheme and a comparison with Crank-Nicolson) This splitting separates the problem into the solution of a quasi-Stokes problem and a nonlinear convection problem. This report describes some details of solution methods used for the two subproblems and results obtained for the driven cavity. The schemes developed for the quasi-Stokes problem are more

(NASA-CR-188008) SOLUTION OF THE NAVIER-STOKES EQUATIONS FOR A DRIVEN CAVITY (Illinois Univ.) 11 p CSCL 20D
N91-20416
Unclas
G3/34 0001581

advanced, at this stage, than those for the nonlinear problem. This report, however, outlines the approaches used for both parts. Future reports will concentrate on the nonlinear problem and more realistic physical examples.

As a model problem we consider a two dimensional square cavity with sides of unit length and a lid moving with unit velocity from left to right. The Navier-Stokes equations are discretized in space on a uniform staggered or MAC mesh. The time discretization is accomplished via the Θ-scheme.

2 The Quasi-Stokes Problem

The linear subproblem encountered with this discretization is called a generalized or quasi Stokes problem and has the form,

$$\alpha u - \mu \Delta u + \nabla p = f \quad (1)$$

$$\nabla \cdot u = 0 \quad (2)$$

where $\alpha = 1/\Delta t$, $\mu = \beta/Re$, and β is a result of splitting the viscous operator. In the above equation, Re is the Reynolds number, p is the pressure, u is the velocity vector, ∇ is the gradient operator, Δ is the vector Laplacian and $\nabla \cdot$

is the divergence operator. In matrix form we have,

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

where A is the discrete form of the elliptic operator $\alpha I - \mu\triangle$. B and B^T are the discrete gradient and negative divergence, respectively. The solution of this linear system can be found by elimination to be,

$$u^* = A^{-1}f \tag{3}$$

$$B^TA^{-1}Bp = B^Tu^* \tag{4}$$

$$u = u^* - A^{-1}Bp \tag{5}$$

The equation for pressure is solved by a preconditioned conjugate gradient method in which the preconditioner involves solving a poisson problem.

It is apparent from this discussion that efficient methods for solving elliptic problems are required. We take advantage of the separability of the resulting equations and use fast Fourier methods to solve the elliptic problems. It is worth noting that such techniques possess ample inherent parallelism.

The solution method just described is known as the conjugate gradient Uzawa scheme,[HaCa 86]. We have conducted some experiments with a new scheme we call the split matrix projection scheme,[Seme 91] Using the same definitions of A, and B above and defining a matrix splitting of A, $A = M - N$, this scheme procedes as follows,

1. initialize Given u_0 and p_0.

2. iterate For $k = 1, 2, \ldots, n$

1. Solve $M\tilde{u_k} = f - Bp_{k-1} + Nu_{k-1}$
2. Project $\tilde{u_k}$ onto $\aleph(B^T)$ (ie. $B^T u_k = 0$)
3. Solve $B^T B p_k = B^T(f - Au_k)$

Numerical experiments have shown this method to be superior to the Uzawa scheme in most cases. Figure 1 shows the performance of this scheme for the cavity problem with a Reynolds number of 500.The time step $\Delta t = 0.1$ and the mesh had 4096 pressure nodes ($h = 1/64$). The test was run on a Sun workstation. The figure shows residual norm vs. time for several splittings. The curves A, B, C and D represent the Jacobi, Gauss-Seidel, approximate factorization and incomplete LU decompositions respectively. The Uzawa scheme required 8.4 seconds to converge for this Reynolds number and time step. The residual norm measured is

$$\|f - Au - Bp\|_2.$$

The speed of the new method is greatly increased for smaller time step and larger Reynolds number. For $\Delta t = 0.01$ all splittings outperform Uzawa at all Reynolds numbers tested. For large Reynolds numbers execution time is reduced by over 50 percent over the Uzawa scheme as is shown in Figure 2. In this case the Uzawa scheme required 6.1 seconds to converge. The definition of convergence for the Uzawa scheme was the time required to achieve,

$$\|f - Au - Bp\|_2 \leq 10^{-9}.$$

Future progress reports will concern the suitability of such stokes solvers on parallel machines.

3 The Nonlinear Problem

The nonlinear problem to be solved has the form,

$$\alpha u + (u \cdot \nabla)u - \mu \triangle u = g$$

where $\alpha = 1/\Delta t$ and $\mu = (1-\beta)/Re$. discretization gives a nonlinear system of equations

$$F(u) = 0$$

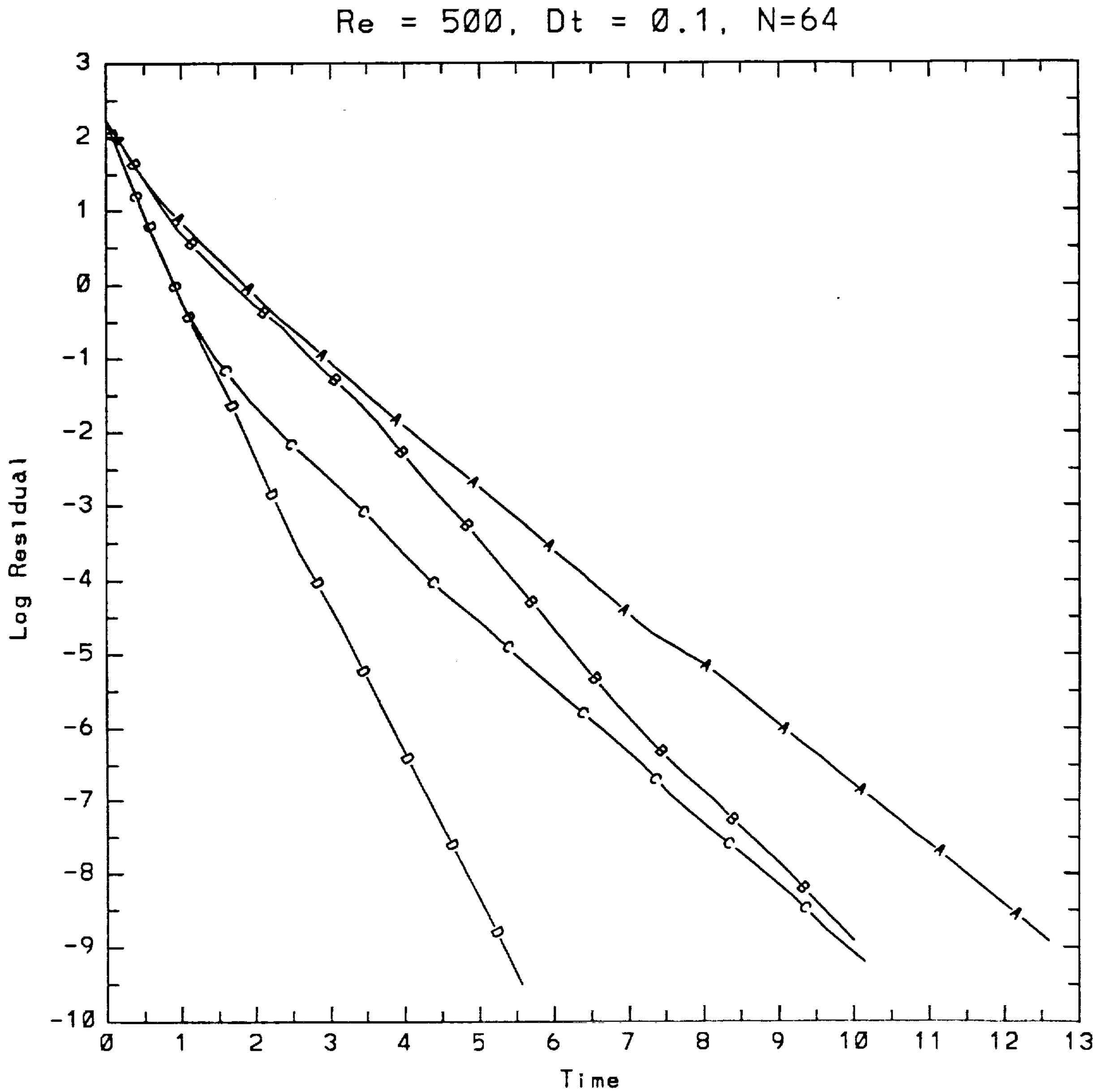

Figure 1: Convergence of Stokes Solvers

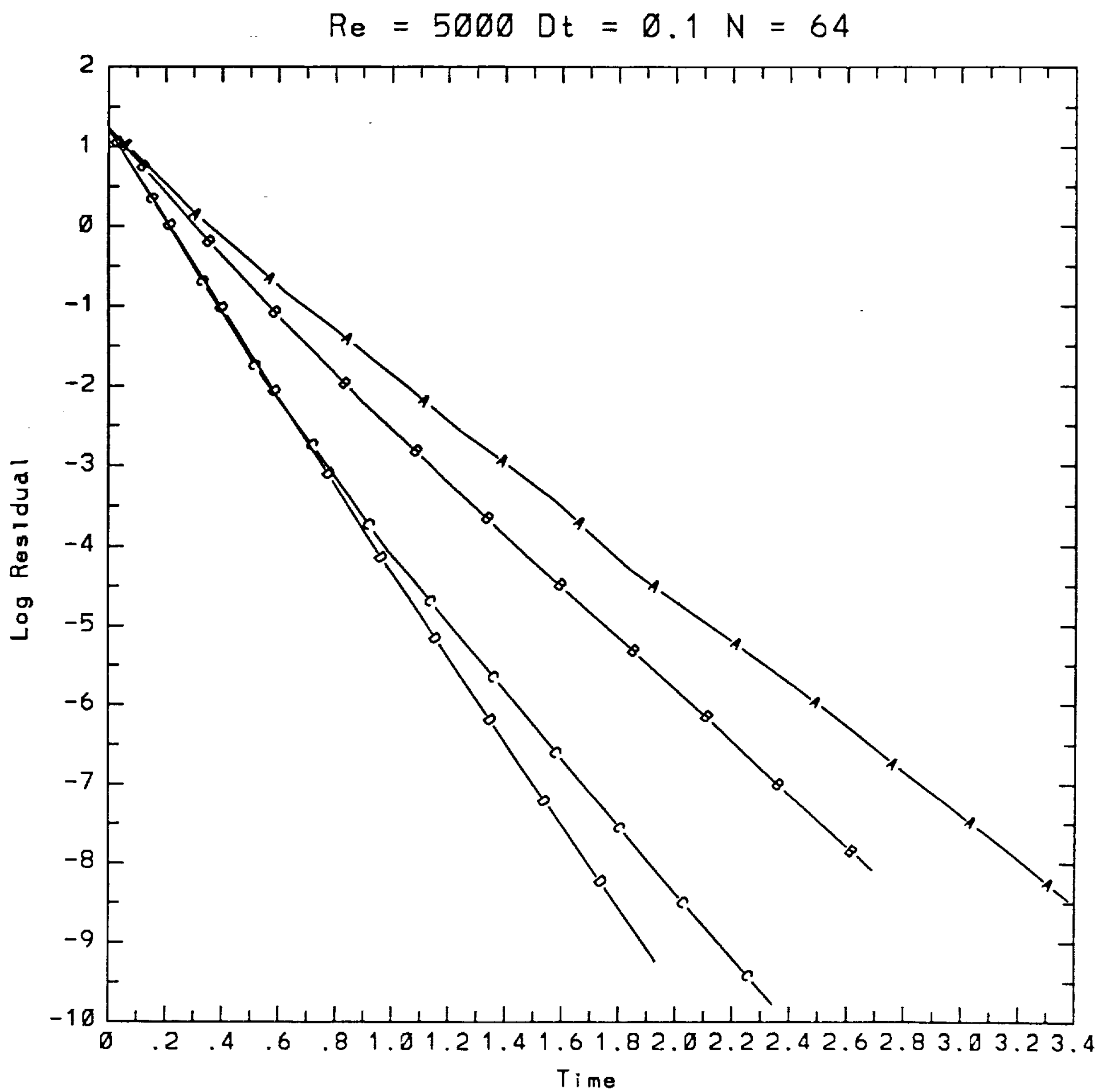

Figure 2: Convergence of Stokes Solvers

where F is a nonlinear function of u. We solve this problem by an inexact or truncated Newton method where the Newton equations,

$$J(u_k)\delta_k = -F(u_k)$$

are solved approximately via an iterative scheme. The solution is updated by

$$u_{k+1} = u_k + \delta_k$$

The Jacobian matrix is not needed explicitly. The reason being the iterative method chosen to solve the linear system (GMRES) needs only a matrix vector product. This can be approximated by a difference quotient of the form,

$$J(u)v \approx \frac{F(u+\sigma v) - F(u)}{\sigma}$$

where σ is a scalar. Note that the function $F(u)$ can be evaluated in parallel. We have found this technique to be an adequate method of solving the non-linear problem. Further research is underway to obtain even more efficient and parallel schemes.

4 Conclusion

Figure 3 shows the steady state vorticity contours in the cavity for a Reynolds number of 500. The overall Navier-Stokes solver described here has been coded on a sequential machine (Sun fileserver). Results obtained so far indicate robustness for both time-accurate and steady state solutions for the above driven cavity problem.

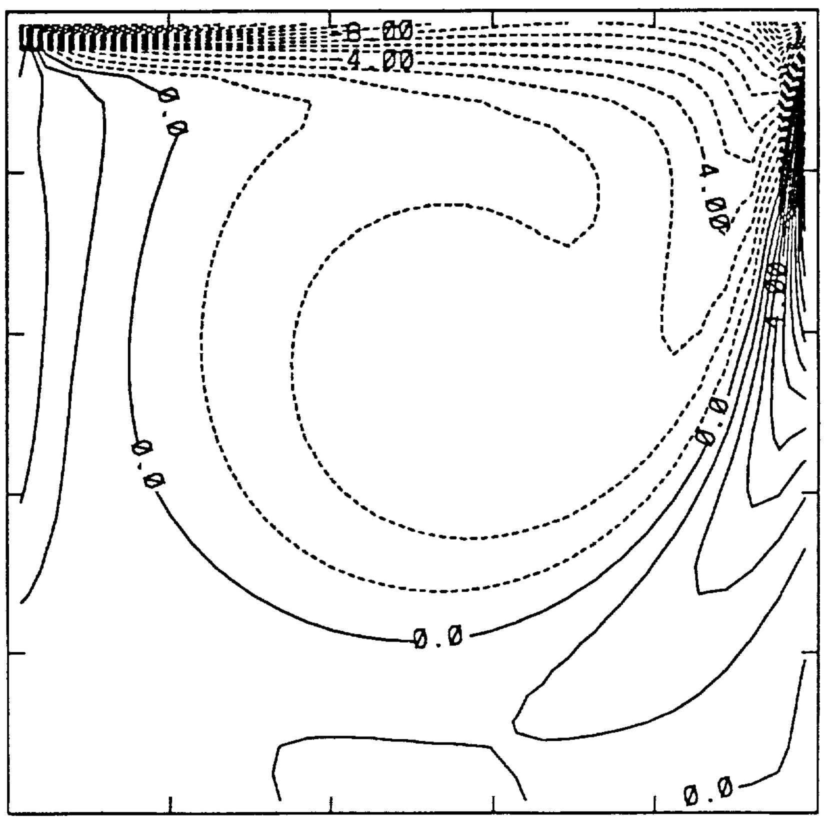

Figure 3: Vorticity Contours

References

[HaCa 86] Hauguel A., Cahouet J., *Finite Element Methods for Incompressible Navier-Stokes and for Shalow Water Equations* Von Karman Institute for Fluid Dynamics, 1986-04, March 3-7, 1986 Rhode Saint Genese Belgium

[Seme 91] Semeraro B.D., *Parallel Operator Splitting Methods for the Navier Stokes Equations* University of Illinois, Urbana Il., 1991 Ph.D. dissertation in preparation.